Suggested solution of Midterm

1. (a) Suppose $\{I_n\}_{n=1}^{\infty}$ is a sequence of closed and bounded interval such that $I_{n+1} \subset I_n$ for all n. Then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

(b) Suppose $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence. Let $M > 0$ such that $|x_n| \leq M$. When $k = 1$, define $a_1 = -M$, $c_1 = M$, $b_1 = (a_1 + c_1)/2$ and $I_1 = [a_1, c_1]$. Suppose we have defined a_k, c_k such that $I_k = [a_k, b_k]$ contains infinity many x_n. Denotes $b_k = (a_k + c_k)/2$. Either $[a_k, b_k]$ or $[b_k, c_k]$ contains infinity many x_n. If $[a_k, b_k]$ does, define $I_{k+1} = [a_k, b_k]$, $a_{k+1} = a_k$ and $c_{k+1} = b_k$. Otherwise, we define $I_{k+1} = [b_k, c_k]$, $a_{k+1} = b_k$ and $c_{k+1} = c_k$.
By construction, I_k is a sequence of nested interval which is bounded and closed. By Nested interval theorem,

$$\{\bar{x}\} = \bigcap_{n=1}^{\infty} I_n.$$

since we have

$$|I_n| = \frac{M}{2^{n-2}} \to 0.$$

On the other hand, each I_k has infinity many element from $\{x_n\}$. Therefore, we can pick a sequence $x_{n_k} \in I_k$. It converges to $\bar{x}$ since

$$|x_{n_k} - \bar{x}| \leq |I_k| \to 0.$$

(c) If $\{x_n\}$ is cauchy, then for all $\epsilon > 0$, there is N such that for all $n, m \geq N$,

$$|x_n - x_m| < \epsilon.$$

In particular, take $\epsilon = 1$, we have for all $n > N_1$,

$$|x_n - x_N| \leq 1.$$

And hence $\{x_k\}$ is a bounded sequence. By above, there is $\bar{x}$ and a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $x_{n_k} \to \bar{x} \in \mathbb{R}$. Hence for $\epsilon > 0$, there is N such that for all $m, k > N$, $(n_m \geq m)$

$$|x_{n_m} - x_k|, |x_{n_m} - \bar{x}| < \epsilon/2.$$

Hence it implies

$$|x_k - \bar{x}| < \epsilon.$$

If $x_n \to \bar{x}$, then for all $\epsilon > 0$, there is N such that for all $n > N$, $|x_n - \bar{x}| < \epsilon/2$. Hence, for all $m, n > N$,

$$|x_n - x_m| \leq |x_n - \bar{x}| + |\bar{x} - x_m| < \epsilon.$$

(a) By Q1, if $\sum_{n=1}^{\infty} |a_n|$ converges, then for all $\epsilon > 0$, there is N such that for $n, m > N$,

$$\sum_{k=n}^{m} |a_k| < \epsilon.$$

Hence, it follows from the triangle inequality that

$$\left| \sum_{k=n}^{m} a_k \right| \leq \sum_{k=n}^{m} |a_k| < \epsilon.$$

By Q1 again (cauchy criterion), $\{\sum_{k=1}^{N} a_k\}_N$ is convergent.

(b) By Q1 (cauchy criterion), For all $\epsilon > 0$, there is N such that for all $m > n > N$, $\sum_{k=n}^{m} y_k < \epsilon$. As $0 \le x_k \le y_k$,
$$\sum_{k=n}^{m} x_k < \epsilon.$$
Thus, $\{s_n = \sum_{k=1}^{n} x_k\}$ is cauchy implying the convergence.

2. (a) Denote $r = (1+a)^{-1}$ where $a > 0$. Using $(1+a)^n \ge 1 + na$, we have
$$r^n = \frac{1}{(1+a)^n} \le \frac{1}{an}.$$
Let $\epsilon > 0$, then for all $n > N = \left[\frac{1}{a\epsilon}\right] + 1$,
$$r^n \le \frac{1}{an} < \epsilon.$$
(b) Denote $r^{\frac{1}{n}} = \frac{1}{1+\sigma_n}$. Then
$$r = \frac{1}{(1+\sigma_n)^n} \le \frac{1}{1+n\sigma_n}.$$
Hence,
$$\sigma_n \le \frac{1-r}{rn}.$$
And hence for $\epsilon > 0$, for all $n > N = \left[\frac{1-r}{r\epsilon}\right] + 1$, we have
$$\left|\frac{1}{1+\sigma_n} - 1\right| = \frac{\sigma_n}{1+\sigma_n} \le \frac{1-r}{rn} < \epsilon.$$

3. (a)
$$\left|\frac{x+1}{x^2-3} - 3\right| = |x-2|\left|\frac{3x+5}{x^2-3}\right|.$$
Let $\epsilon > 0$ be given, then we may choose $\delta = \min\{0.1, \epsilon/100\}$. Then for all $0 < |x-2| < \delta$,
$$|x-2|\left|\frac{3x+5}{x^2-3}\right| \le 100|x-2| < \epsilon.$$
(b) Let $M > 0$, pick $\delta = \min\{1, 5M^{-1}\}$. Then for all $3 - \delta < x < 3$,
$$\frac{x^2+1}{x-3} \le \frac{5}{x-3} < -M.$$

4. By assumption, take $\epsilon = 1$, we obtain δ_1 so that for all $x \in A$ where $0 < |x-c| < \delta_1$, for $i = 1, 2$,
$$|f_i(x)| \le |f_i(x) - l_i| + |l_i| < |l_i| + 1.$$
Denote $M = |l_1| + |l_2| + 2$. For $\epsilon > 0$, there is $\delta_2 = \delta_2(\epsilon, M)$ such that for all $x \in A$ where $0 < |x-c| < \delta_2$, we have
$$|f_i(x) - l_i| < \frac{\epsilon}{4M}.$$
Hence, for the same $\epsilon > 0$, if $x \in A$ where $0 < |x-c| < \min\{\delta_1, \delta_2\}$, we have
$$\begin{aligned}|f_1 f_2 - l_1 l_2| &\le |f_2(x)||f_1(x) - l_1| + |l_1||f_2(x) - l_2| \\ &\le M|f_1(x) - l_1| + M|f_2(x) - l_2| \\ &\le \frac{\epsilon}{2}.\end{aligned}$$